

7.1. Locally symmetric spaces. Let M be a connected m -dimensional Riemannian manifold. Then M is called *locally symmetric* if for all $p \in M$ there is a normal neighborhood $B(p, r)$ such that the *local geodesic reflection*

$$\sigma_p := \exp_p \circ (-\text{id}) \circ \exp_p^{-1} : B(p, r) \rightarrow B(p, r)$$

is an isometry.

- (a) Show that if M is locally symmetric, then $DR \equiv 0$.

Hint: Use that $d(\sigma_p)_p = -\text{id}$ on $T_p M$.

- (b) Suppose that $DR \equiv 0$. Show that if $c : [-1, 1] \rightarrow M$ is a geodesic and $\{E_i\}_{i=1}^m$ is a parallel orthonormal frame along c , then

$$R(E_i, c')c' = \sum_{k=1}^m r_i^k E_k$$

for constants r_i^k .

- (c) Show that if $DR \equiv 0$, then M is locally symmetric.

Hint: Let $q \in B(p, r)$, $q \neq p$, and $v \in T_q M$. To show that $|d(\sigma_p)_q(v)| = |v|$, consider the geodesic $c : [-1, 1] \rightarrow B(p, r)$ with $c(0) = p$, $c(1) = q$, and a Jacobi field Y along c with $Y(0) = 0$, $Y(1) = v$. Use (b).

7.2. Conjugate points in manifolds with curvature bounded from above.

- (a) Prove directly, without using the Rauch Comparison Theorem, that there are no conjugate points in manifolds with non-positive sectional curvature.
- (b) Show that in manifolds with sectional curvature at most κ , where $\kappa > 0$, there are no conjugate points along geodesics of length $< \pi/\sqrt{\kappa}$.
- (c) Show that if $c : [0, \pi/\sqrt{\kappa}] \rightarrow M$ is a unit speed geodesic in a manifold with $\text{sec} \geq \kappa > 0$, then some $c(t)$ is conjugate to $c(0)$ along $c|_{[0,t]}$.

7.3. Volume comparison. Let M be an m -dimensional Riemannian manifold with sectional curvature $\text{sec} \leq \kappa$, $p \in M$, and $r > 0$ such that $\exp_p|_{B_r(0)}$ is a diffeomorphism. Let $V_\kappa^m(r)$ denote the volume of a ball of radius r in the m -dimensional model space M_κ^m . Prove that:

$$V(B_r(p)) \geq V_\kappa^m(r).$$